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Discrete Mathematics 308 (2008) 4724–4733

DISCRETE MATHEMATICS

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Congruence kernels of orthomodular implication algebras

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Received 8 June 2006; received in revised form 21 August 2007; accepted 23 August 2007
Available online 22 October 2007

Abstract

Abstracting from certain properties of the implication operation in Boolean algebras leads to so-called orthomodular implication algebras. These are in a natural one-to-one correspondence with families of orthomodular lattices. It is proved that congruence kernels of orthomodular implication algebras are in a natural one-to-one correspondence with families of compatible p-filters on the corresponding orthomodular lattices.

MSC: 08A30; 20N02; 06A12; 06C15

Keywords: Orthomodular implication algebra; Weak regularity; Permutability at 1; 3-Permutability; Orthomodular lattice; Orthomodular join-semilattice; Congruence; Compatible congruence family; Congruence kernel; p-Filter; Compatible filter family

In the literature many attempts were made in order to investigate properties of the implication operation in generalizations of Boolean algebras. These attempts led to different types of so-called implication algebras (cf. e.g. [1,2,5,6]). It is interesting to note that these types of implication algebras are in a natural one-to-one correspondence with join-semilattices with 1 the principal filters of which are certain generalizations of Boolean algebras. Hence the question arises if there is a natural one-to-one correspondence between congruence kernels of these implication algebras on the one side and certain families of congruence kernels of the corresponding generalizations of Boolean algebras on the other side. We solve this problem for so-called orthomodular implication algebras introduced in [5]. Corresponding results concerning orthoimplication algebras are contained in [7].

In [5], orthomodular implication algebras were introduced as algebras reflecting properties of a certain implication operation in orthomodular lattices (which are generalizations of Boolean algebras):

Definition 1 (cf. *Chajda et al. [5,6]*). An *orthomodular implication algebra* is an algebra $(A, \cdot, 1)$ of type $(2, 0)$ satisfying

$$xx = 1,$$

$$(xy)x = x,$$

[1] Support of the research of the first two authors by the Czech Government Research Project MSM 6198959214 is gratefully acknowledged.
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doi:10.1016/j.disc.2007.08.101

$$(xy)y = (yx)x,$$

$$(((xy)y)z)(xz) = 1$$

and

$$((((((xy)y)z)x)x)z)x)x = (((xy)y)z)z.$$

Remark 1. In every orthomodular implication algebra it holds $1x = x$ and $x1 = 1$ since $1x = (xx)x = x$ and $x1 = (1x)1 = 1$.

First we want to prove some congruence properties for the variety of orthomodular implication algebras. For any algebra $\mathscr{B}$ let $\operatorname{Con}\mathscr{B}$ denote the set of all congruences on $\mathscr{B}$.

Definition 2. Let $\mathscr{A}$ be an algebra with 1. $\mathscr{A}$ is called *weakly regular* if $\Theta, \Phi \in \operatorname{Con}\mathscr{A}$ and $[1]\Theta = [1]\Phi$ together imply $\Theta = \Phi$. $\mathscr{A}$ is called *permutable at* 1 if $[1](\Theta \circ \Phi) = [1](\Phi \circ \Theta)$ for all $\Theta, \Phi \in \operatorname{Con}\mathscr{A}$. $\mathscr{A}$ is called 3-*permutable* if $\Theta \circ \Phi \circ \Theta = \Phi \circ \Theta \circ \Phi$ for all $\Theta, \Phi \in \operatorname{Con}\mathscr{A}$.

Theorem 1. *The variety $\mathscr{V}$ of orthomodular implication algebras is weakly regular, permutable at* 1 *and* 3-*permutable.*

Proof. Put $t_1(x, y) := xy$ and $t_2(x, y) := yx$. Then $t_1(x, x) = t_2(x, x) = 1$. Conversely, if $t_1(x, y) = t_2(x, y) = 1$ then $x = 1x = (yx)x = (xy)y = 1y = y$. Hence $\mathscr{V}$ is weakly regular according to Theorem 6.4.3 in [4]. If $t(x, y) := yx$ then $t(x, x) = 1$ and $t(x, 1) = x$ and hence $\mathscr{V}$ is permutable at 1 due to Theorem 6.6.11 in [4]. Finally, if $t_1(x, y, z) := (zy)x$ and $t_2(x, y, z) := (xy)z$ then

$$t_1(x, z, z) = (zz)x = 1x = x,$$

$$t_1(x, x, z) = (zx)x = (xz)z = t_2(x, z, z)$$

and

$$t_2(x, x, z) = (xx)z = 1z = z$$

and hence $\mathscr{V}$ is 3-permutable according to Theorem 3.1.18 in [4]. $\square$

In [5], a natural one-to-one correspondence between orthomodular implication algebras and certain families of orthomodular lattices was established. In order to be able to define these structures we first need the definition of an orthomodular lattice. (For the theory of orthomodular lattices we refer the reader to the monographs [8,3,9].)

Definition 3. An *orthomodular lattice* is an algebra $(L, \vee, \wedge, ', 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(L, \vee, \wedge, 0, 1)$ is a bounded lattice and

$$x \vee x' = 1,$$

$$x \wedge x' = 0,$$

$$(x \vee y)' = x' \wedge y',$$

$$(x \wedge y)' = x' \vee y',$$

$$(x')' = x$$

and

$$x \leqslant y \text{ implies } y = x \vee (y \wedge x').$$

The third and fourth condition are the well-known De Morgan laws and the last condition is the so-called orthomodular law.

Now we are able to define the order-theoretical counterpart of orthomodular implication algebras introduced in [5]:

Definition 4 (cf. *Chajda et al. [5]*). An *orthomodular join-semilattice* is a partial algebra $\mathscr{S} = (A, \vee, ({}^x; x \in A), 1)$ such that $(A, \vee, 1)$ is a join-semilattice with 1 and for each $x \in A$, ^{x} is a unary operation on $[x, 1]$ such that $([x, 1], \vee, \wedge_x, {}^x, x, 1)$ is an orthomodular lattice.

In the following let $\mathscr{S} = (A, \vee, ({}^x; x \in A), 1)$ be an arbitrary, but fixed orthomodular join-semilattice with 1.

Remark 2. It should be remarked that $\wedge_x$ does not depend on x. More precisely, this means the following: Let $(A, \leqslant)$ denote the partially ordered set corresponding to the join-semilattice $(A, \vee)$ and $a, b \in A$. Then $a \wedge b$ exists in $(A, \leqslant)$ if and only if a and b have a common lower bound in $(A, \leqslant)$. Moreover, for every common lower bound c of a and b it holds $a \wedge_c b = a \wedge b$. This can be seen as follows: If c and d are common lower bounds of a and b then a and b are common upper bounds of c and d and hence $c \vee d \leqslant a, b$. Therefore $c \vee d$ is a common lower bound of a and b in $([c, 1], \leqslant)$ and hence $d \leqslant c \vee d \leqslant a \wedge_c b$. Therefore $a \wedge_c b = a \wedge b$ and the index in "$\wedge_c$" can be deleted.

Remark 3. According to the De Morgan laws $x \wedge y = (x^a \vee y^a)^a$ holds for all $a \in A$ and $x, y \in [a, 1]$.

The natural one-to-one correspondence between orthomodular implication algebras and orthomodular join-semilattices can be formulated as follows (for other types of implication algebras and their corresponding order-theoretical counterparts cf. e.g. [1,2]):

Theorem 2 (cf. *Chajda et al. [5]*). *For every fixed set A the formulas*

$$x \vee y = (xy)y,$$

$$x^y = xy$$

and

$$xy = (x \vee y)^y$$

induce mutually inverse bijections between the set of all orthomodular implication algebras over A and the set of all orthomodular join-semilattices over A.

In the following let $\mathscr{A} = (A, \cdot, 1)$ be an arbitrary, but fixed orthomodular implication algebra and $\mathscr{S}(\mathscr{A}) = (A, \vee, ({}^x; x \in A), 1)$ its corresponding orthomodular join-semilattice.

For orthomodular join-semilattices $\mathscr{S}$ we need a certain notion corresponding to the notion of a congruence.

Definition 5. A *compatible congruence family* on $\mathscr{S}$ is a family $(\Theta_x; x \in A)$ of congruences Θ_x on $([x, 1], \vee, \wedge, {}^x, x, 1)$ (as orthomodular lattices) such that $\Theta_y = \Theta_x \cap [y, 1]^2$ for all $x, y \in A$ with $x \leqslant y$ and $(z^x, z^y) \in \Theta_x$ for all $x, y, z \in A$ satisfying both $x \leqslant y \leqslant z$ and $(x, y) \in \Theta_x$. Let $\mathrm{CCF}(\mathscr{S})$ denote the set of all compatible congruence families on $\mathscr{S}$. On $\mathrm{CCF}(\mathscr{S})$ we define a binary relation $\leqslant$ by

$$(\Theta_x; x \in A) \leqslant (\Phi_x; x \in A) \text{ if } \Theta_x \subseteq \Phi_x \quad \text{for all } x \in A.$$

Remark 4. $(\mathrm{CCF}(\mathscr{S}), \leqslant)$ is a complete lattice.

Now we can formulate the natural one-to-one correspondence between congruences on $\mathscr{A}$ and compatible congruence families on $\mathscr{S}(\mathscr{A})$:

Theorem 3. *The formulas*

$$\Theta_x = \Theta \cap [x, 1]^2$$

and

$$\Theta = \{(x, y) \in A^2 \mid (x, x \vee y) \in \Theta_x \textit{ and } (x \vee y, y) \in \Theta_y\}$$

induce mutually inverse isomorphisms between $(\mathrm{Con}\,\mathscr{A}, \subseteq)$ *and* $(\mathrm{CCF}(\mathscr{S}(\mathscr{A})), \leqslant)$.

Proof. Let $a, b, c, d \in A$. If $\Theta \in \mathrm{Con}\,\mathscr{A}$ and $\Theta_x := \Theta \cap [x, 1]^2$ for all $x \in A$ then

$$b \vee c = (bc)c,$$

$$b \wedge c = (((ba)(ca))(ca))a$$

and

$$b^a = ba$$

if $b, c \geqslant a$ showing that $\Theta_a \in \mathrm{Con}([a, 1], \vee, \wedge, {}^a, a, 1)$. Clearly,

$$\Theta_b = \Theta \cap [b, 1]^2 = \Theta \cap ([a, 1]^2 \cap [b, 1]^2) = (\Theta \cap [a, 1]^2) \cap [b, 1]^2 = \Theta_a \cap [b, 1]^2$$

if $a \leqslant b$. Finally, $(c^a, c^b) = (ca, cb) \in \Theta_a$ provided both $a \leqslant b \leqslant c$ and $(a, b) \in \Theta_a$. This shows $(\Theta_x; x \in A) \in \mathrm{CCF}(\mathscr{S}(\mathscr{A}))$. Moreover, the following are equivalent:

$$(a, a \vee b) \in \Theta_a \quad \text{and} \quad (a \vee b, b) \in \Theta_b,$$

$$(a, a \vee b), (a \vee b, b) \in \Theta$$

and

$$(a, b) \in \Theta.$$

(Observe that $a \vee b = (ab)b$.)

Conversely, assume $(\Theta_x; x \in A) \in \mathrm{CCF}(\mathscr{S}(\mathscr{A}))$ and define

$$\Theta := \{(x, y) \in A^2 \mid (x, x \vee y) \in \Theta_x \text{ and } (x \vee y, y) \in \Theta_y\}.$$

Clearly, Θ is reflexive and symmetric.

In the sequel we frequently use the following fact:

$$\text{If } a, b \leqslant c, d \text{ and } (c, d) \in \Theta_a \quad \text{then } (c, d) \in \Theta_b$$

which follows from

$$\Theta_a \cap [a \vee b, 1]^2 = \Theta_{a \vee b} = \Theta_b \cap [a \vee b, 1]^2 \subseteq \Theta_b.$$

If $(a, b), (b, c) \in \Theta$ then

$$a \vee b \vee c = (a \vee b) \vee (b \vee c) \Theta_b (a \vee b) \vee b = a \vee b$$

and hence

$$a \vee b \vee c \Theta_a a \vee b.$$

This implies

$$a \vee c = a \vee (a \vee c) \Theta_a (a \vee b) \vee (a \vee c) = a \vee b \vee c \Theta_a a \vee b \Theta_a a$$

and

$$a \vee b \vee c = (a \vee b) \vee (b \vee c)\Theta_b b \vee (b \vee c) = b \vee c,$$

hence

$$a \vee b \vee c\Theta_c b \vee c$$

which implies

$$a \vee c = (a \vee c) \vee c\Theta_c(a \vee c) \vee (b \vee c) = a \vee b \vee c\Theta_c b \vee c\Theta_c c,$$

showing $(a, c) \in \Theta$ and proving transitivity of Θ.

Next we show that Θ is a right congruence on $\mathscr{A}$. Assume $(a, b) \in \Theta$. Then

$$a \vee b \vee c = (a \vee b) \vee (a \vee c)\Theta_a a \vee (a \vee c) = a \vee c$$

and hence $a \vee b \vee c\Theta_c a \vee c$. Moreover,

$$a \vee b \vee c = (a \vee b) \vee (b \vee c)\Theta_b b \vee (b \vee c) = b \vee c$$

and hence $a \vee b \vee c\Theta_c b \vee c$. Together we obtain

$$a \vee c\Theta_c a \vee b \vee c\Theta_c b \vee c$$

and therefore

$$ac = (a \vee c)^c\Theta_c(b \vee c)^c = bc.$$

Now

$$ac \vee bc\Theta_c bc \vee bc = bc \text{ implies } ac \vee bc\Theta_{bc} bc$$

and

$$ac \vee bc\Theta_c ac \vee ac = ac \text{ implies } ac \vee bc\Theta_{ac} ac.$$

Together this shows $(ac, bc) \in \Theta$ proving that Θ is a right congruence on $\mathscr{A}$.

From this it follows that $(a, b) \in \Theta$ implies $(a \vee c, b \vee c) \in \Theta$ since $(a, b) \in \Theta$ implies $(ac, bc) \in \Theta$ and hence $(a \vee c, b \vee c) = ((ac)c, (bc)c) \in \Theta$.

Next we show that Θ is a left congruence on $\mathscr{A}$. Assume $(a, b) \in \Theta$. Then

$$a \vee b \vee c = (a \vee b) \vee (a \vee c)\Theta_a a \vee (a \vee c) = a \vee c$$

and hence

$$(a \vee b \vee c)^a\Theta_a(a \vee c)^a$$

which implies

$$(a \vee b \vee c)^a\Theta_{(a \vee b \vee c)^a}(a \vee c)^a.$$

Thus

$$(a \vee b \vee c)^a \vee (a \vee c)^a = (a \vee c)^a\Theta_{(a \vee b \vee c)^a}(a \vee b \vee c)^a$$

and

$$(a \vee b \vee c)^a \vee (a \vee c)^a = (a \vee c)^a\Theta_{(a \vee c)^a}(a \vee c)^a$$

showing $((a\vee b\vee c)^a, (a\vee c)^a)\in\Theta$. Analogously, it follows $((a\vee b\vee c)^b, (b\vee c)^b)\in\Theta$. Now $a\leqslant a\vee b\leqslant a\vee b\vee c$ and $(a, a\vee b)\in\Theta_a$ together imply $((a\vee b\vee c)^a, (a\vee b\vee c)^{a\vee b})\in\Theta_a$. From this it follows

$$\begin{aligned}&((a\vee b\vee c)^a\vee b, (a\vee b\vee c)^a\vee(a\vee b\vee c)^{a\vee b}\vee b)\\&\quad=((a\vee b\vee c)^a\vee(a\vee b\vee c)^a\vee b, (a\vee b\vee c)^{a\vee b}\vee(a\vee b\vee c)^a\vee b)\in\Theta_a\end{aligned}$$

and hence

$$((a\vee b\vee c)^a\vee b, (a\vee b\vee c)^a\vee(a\vee b\vee c)^{a\vee b}\vee b)\in\Theta_{(a\vee b\vee c)^a\vee b}.$$

Analogously, it follows

$$\begin{aligned}&((a\vee b\vee c)^a\vee b\vee(a\vee b\vee c)^{a\vee b}, (a\vee b\vee c)^{a\vee b})\\&\quad=((a\vee b\vee c)^a\vee a\vee b\vee(a\vee b\vee c)^{a\vee b}, (a\vee b\vee c)^{a\vee b}\vee a\vee b\vee(a\vee b\vee c)^{a\vee b})\in\Theta_a\end{aligned}$$

and hence

$$((a\vee b\vee c)^a\vee b\vee(a\vee b\vee c)^{a\vee b}, (a\vee b\vee c)^{a\vee b})\in\Theta_{(a\vee b\vee c)^{a\vee b}}.$$

This shows $((a\vee b\vee c)^a\vee b, (a\vee b\vee c)^{a\vee b})\in\Theta$. Analogously, it follows $((a\vee b\vee c)^b\vee a, (a\vee b\vee c)^{a\vee b})\in\Theta$. Due to transitivity of Θ this implies $((a\vee b\vee c)^a\vee b, (a\vee b\vee c)^b\vee a)\in\Theta$. Now we have

$$\begin{aligned}&ca=(c\vee a)^a=(a\vee c)^a\Theta(a\vee b\vee c)^a=(a\vee b\vee c)^a\vee a\Theta(a\vee b\vee c)^a\vee b\\&\qquad\Theta(a\vee b\vee c)^b\vee a\Theta(a\vee b\vee c)^b\vee b=(a\vee b\vee c)^b\Theta(b\vee c)^b=(c\vee b)^b=cb.\end{aligned}$$

Because of transitivity of Θ, $(ca, cb)\in\Theta$ proving that Θ is a left congruence on $\mathscr{A}$.

Hence $\Theta\in\operatorname{Con}\mathscr{A}$. Moreover, the following are equivalent:

$$(b, c)\in\Theta\cap[a, 1]^2,$$

$$b, c\geqslant a, (b, b\vee c)\in\Theta_b \quad\text{and}\quad (b\vee c, c)\in\Theta_c,$$

$$(b, b\vee c), (b\vee c, c)\in\Theta_a$$

and

$$(b, c)\in\Theta_a.$$

This shows that the mappings induced by the formulas stated in the theorem are mutually inverse bijections between $\operatorname{Con}\mathscr{A}$ and $\mathrm{CCF}(\mathscr{S}(\mathscr{A}))$. That they are in fact isomorphisms between $(\operatorname{Con}\mathscr{A}, \subseteq)$ and $(\mathrm{CCF}(\mathscr{S}(\mathscr{A})), \leqslant)$, is evident. □

Corollary 1. *Every congruence on $\mathscr{A}$ is uniquely determined by its restrictions to the intervals $[x, 1]$, $x\in A$.*

How congruences Θ on an orthomodular implication algebra are determined by their kernels $[1]\,\Theta$ is described by the following:

Lemma 1. $\Theta=\{(x, y)\in A^2\mid xy, yx\in[1]\Theta\}$ *for all* $\Theta\in\operatorname{Con}\mathscr{A}$.

Proof. If $(a, b)\in\Theta$ then $ab, ba\in[aa]\Theta=[1]\Theta$ and if, conversely, $a, b\in A$ and $ab, ba\in[1]\Theta$ then $a=1a\Theta(ba)a=(ab)b\Theta 1b=b$ and hence $(a, b)\in\Theta$. □

Now we introduce the notion of a congruence kernel of an orthomodular implication algebra.

Definition 6. A subset F of A is called a *congruence kernel* of $\mathscr{A}$ if there exists a congruence Θ on $\mathscr{A}$ with $[1]\Theta = F$. Let $\mathrm{CK}(\mathscr{A})$ denote the set of all congruence kernels of $\mathscr{A}$.

Remark 5. $(\mathrm{CK}(\mathscr{A}), \subseteq)$ is a complete lattice.

The natural one-to-one correspondence between congruences on orthomodular implication algebras and their kernels is established by the following:

Theorem 4. *The formulas*

$$F = [1]\Theta$$

and

$$\Theta = \{(x, y) \in A^2 \mid xy, yx \in F\}$$

induce mutually inverse isomorphisms between $(\mathrm{Con}\,\mathscr{A}, \subseteq)$ *and* $(\mathrm{CK}(\mathscr{A}), \subseteq)$.

Proof. The proof follows from Lemma 1. □

In the following let $\mathscr{L} = (L, \vee, \wedge, ', 0, 1)$ be an arbitrary, but fixed orthomodular lattice.

Definition 7 (cf. *Kalmbach [8]*). Two elements a, b of L are called *perspective* to each other (which is denoted by $a \sim b$) if they have a common lattice-theoretical complement, i.e. if there exists an element c of L satisfying $a \vee c = b \vee c = 1$ and $a \wedge c = b \wedge c = 0$. A subset F of L is called a *p-filter* of $\mathscr{L}$ if it is a filter of $\mathscr{L}$ that is closed with respect to perspectivity, i.e. if F satisfies

(i) $F \neq \emptyset$.
(ii) $x, y \in F$ implies $x \wedge y \in F$.
(iii) $x \in F$, $y \in L$ and $x \leqslant y$ together imply $y \in F$.
(iv) $x \in F$, $y \in L$ and $x \sim y$ together imply $y \in F$.

Let $\mathrm{F}(\mathscr{L})$ denote the set of all p-filters of $\mathscr{L}$.

It is well known that p-filters of orthomodular lattices can be characterized in the following way:

Theorem 5 (cf. *Kalmbach [8]*). *A subset F of L is a p-filter of $\mathscr{L}$ if and only if there exists a congruence Θ on $\mathscr{L}$ satisfying* $[1]\Theta = F$.

Remark 6. $(\mathrm{F}(\mathscr{L}), \subseteq)$ is a complete lattice.

The following one-to-one correspondence between congruences on and p-filters of orthomodular lattices (generalizing the corresponding one for Boolean algebras) is well known:

Theorem 6 (cf. *Kalmbach [8]*). *The formulas*

$$F = [1]\Theta$$

and

$$\Theta = \{(x, y) \in L^2 \mid (x \wedge y) \vee (x' \wedge y') \in F\}$$

induce mutually inverse isomorphisms between $(\mathrm{Con}\,\mathscr{L}, \subseteq)$ *and* $(\mathrm{F}(\mathscr{L}), \subseteq)$.

For describing congruence kernels of orthomodular implication algebras by p-filters of the corresponding orthomodular lattices we need the following concept:

Definition 8. A *compatible filter family* on $\mathscr{S}$ is a family $(F_x; x \in A)$ of p-filters F_x of $([x, 1], \vee, \wedge, ^x, x, 1)$ such that for all $x, y, z, u \in A$ with $x \leqslant y \leqslant z, u$ the conditions (i) and (ii) hold:

(i) $y^x \in F_x$ implies $(z^x \wedge z^y) \vee (z \wedge (z^y)^x) \in F_x$ and
(ii) $(z \wedge u) \vee (z^x \wedge u^x) \in F_x$ is equivalent to $(z \wedge u) \vee (z^y \wedge u^y) \in F_y$.

Let $\mathrm{CFF}(\mathscr{S})$ denote the set of all compatible filter families on $\mathscr{S}$. On $\mathrm{CFF}(\mathscr{S})$ we define a binary relation $\leqslant$ by

$$(F_x; x \in A) \leqslant (G_x; x \in A) \text{ if } F_x \subseteq G_x \quad \text{for all } x \in A.$$

Remark 7 $(CFF\ (\mathscr{S})), \leqslant)$. is a complete lattice.

In the proof of the next theorem we need the following easy property of congruence kernels of orthomodular implication algebras:

Lemma 2. *If* $F \in \mathrm{CK}(\mathscr{A})$, $a \in F$, $b \in A$ *and* $a \leqslant b$ *then* $b \in F$.

Proof. If $\Theta \in \mathrm{Con}\,\mathscr{A}$ with $[1]\Theta = F$ then $b = a \vee b \in [1 \vee b]\Theta = [1]\Theta = F$. $\square$

Now we are able to formulate and prove the natural one-to-one correspondence between congruence kernels of orthomodular implication algebras and compatible filter families on the corresponding orthomodular join-semilattices.

Theorem 7. *The formulas*

$$F_x = F \cap [x, 1]$$

and

$$F = \bigcup_{x \in A} F_x$$

induce mutually inverse isomorphisms between $(\mathrm{CK}(\mathscr{A}), \subseteq)$ *and* $(\mathrm{CFF}(\mathscr{S}(\mathscr{A})), \leqslant)$.

Proof. Let $a, b, c, d \in A$. First assume $F \in \mathrm{CK}(\mathscr{A})$ and put $F_x := F \cap [x, 1]$ for all $x \in A$. Then there exists a congruence Θ on $\mathscr{A}$ with $[1]\Theta = F$. Now $\Theta \cap [a, 1]^2 \in \mathrm{Con}([a, 1], \vee, \wedge, ^a, a, 1)$ according to Theorem 3 and

$$F_a = F \cap [a, 1] = [1]\Theta \cap [a, 1] = [1](\Theta \cap [a, 1]^2) \in \mathrm{F}([a, 1], \vee, \wedge, ^a, a, 1).$$

If $a \leqslant b \leqslant c$ and $b^a \in F_a = [1]\Theta \cap [a, 1]$ then

$$b = b \vee a = (ba)a = b^a a \Theta 1a = a$$

and hence

$$\begin{aligned}(c^a \wedge c^b) \vee (c \wedge (c^b)^a) &= ((ca) \wedge (cb)) \vee (c \wedge ((cb)a)) \\ &\in [((ca) \wedge (ca)) \vee (c \wedge ((ca)a))]\Theta \cap [a, 1] \\ &= [(ca) \vee (c \wedge (c \vee a))]\Theta \cap [a, 1] = [c^a \vee c]\Theta \cap [a, 1] \\ &= [1]\Theta \cap [a, 1] = F_a.\end{aligned}$$

Finally, if $a \leqslant b \leqslant c, d$ then according to Theorem 6 the following are equivalent:

$$(c \wedge d) \vee (c^a \wedge d^a) \in F_a,$$

$$(c, d) \in \Theta \cap [a, 1]^2,$$

$$(c, d) \in \Theta \cap [b, 1]^2$$

and

$$(c \wedge d) \vee (c^b \wedge d^b) \in F_b.$$

This shows $(F_x; x \in A) \in \mathrm{CFF}(\mathscr{S}(\mathscr{A}))$. Moreover,

$$\bigcup_{x \in A} F_x = \bigcup_{x \in A} (F \cap [x, 1]) = F \cap \bigcup_{x \in A} [x, 1] = F \cap A = F.$$

Conversely, assume $(F_x; x \in A) \in \mathrm{CFF}(\mathscr{S}(\mathscr{A}))$ and put $F := \bigcup_{x \in A} F_x$. Then for every $x \in A$ there exists a congruence Θ_x on $([x, 1], \vee, \wedge, {}^x, x, 1)$ with $[1]\Theta_x = F_x$. If $a \leqslant b$ then according to Theorem 6 the following are equivalent:

$$(c, d) \in \Theta_b,$$

$$c, d \geqslant b \quad \text{and} \quad (c \wedge d) \vee (c^b \wedge d^b) \in F_b,$$

$$c, d \geqslant b \quad \text{and} \quad (c \wedge d) \vee (c^a \wedge d^a) \in F_a$$

and

$$(c, d) \in \Theta_a \cap [b, 1]^2.$$

Therefore $\Theta_b = \Theta_a \cap [b, 1]^2$ if $a \leqslant b$ and hence $(\Theta_x; x \in A) \in \mathrm{CCF}(\mathscr{S}(\mathscr{A}))$. Put

$$\Theta := \{(x, y) \in A^2 \mid (x, x \vee y) \in \Theta_x \text{ and } (x \vee y, y) \in \Theta_y\}.$$

According to Theorem 3, $\Theta \in \mathrm{Con}\,\mathscr{A}$ and $\Theta \cap [x, 1]^2 = \Theta_x$ for all $x \in A$. Now

$$\begin{aligned} F &= \bigcup_{x \in A} F_x = \bigcup_{x \in A} ([1]\Theta_x) = \bigcup_{x \in A} ([1](\Theta \cap [x, 1]^2)) = \bigcup_{x \in A} ([1]\Theta \cap [x, 1]) \\ &= [1]\Theta \cap \bigcup_{x \in A} [x, 1] = [1]\Theta \in \mathrm{CK}(\mathscr{A}). \end{aligned}$$

Moreover,

$$\begin{aligned} F \cap [a, 1] &= \left(\bigcup_{x \in A} F_x \right) \cap [a, 1] = \bigcup_{x \in A} (F_x \cap [a, 1]) = \bigcup_{x \in A} ((F_x \cap [x, 1]) \cap [a, 1]) \\ &= \bigcup_{x \in A} (F_x \cap ([x, 1] \cap [a, 1])) = \bigcup_{x \in A} (F_x \cap [a \vee x, 1]) = \bigcup_{x \in A} F_{a \vee x} \\ &= \bigcup_{x \in A} (F_a \cap [a \vee x, 1]) = F_a \cap \bigcup_{x \in A} [a \vee x, 1] = F_a \cap [a, 1] = F_a. \end{aligned}$$

The rest of the proof is clear. $\square$

From Theorem 7 we deduce the following nice characterization of congruence kernels of orthomodular implication algebras:

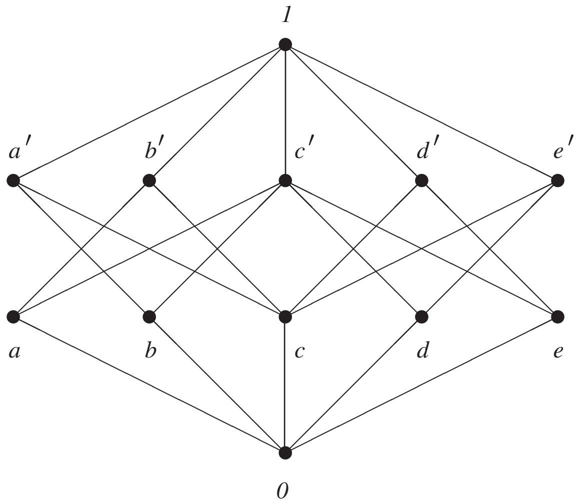
Corollary 2. *A subset F of A is a congruence kernel of $\mathscr{A}$ if and only if for all $x, y, z, u \in A$ with $x \leqslant y \leqslant z, u$ the conditions* (i)–(iii) *are satisfied*:

(i) $F \cap [x, 1] \in \mathrm{F}([x, 1], \vee, \wedge, {}^{x}, x, 1)$,
(ii) $y^{x} \in F$ *implies* $(z^{x} \wedge z^{y}) \vee (z \wedge (z^{y})^{x}) \in F$ *and*
(iii) $(z \wedge u) \vee (z^{x} \wedge u^{x}) \in F$ *is equivalent to* $(z \wedge u) \vee (z^{y} \wedge u^{y}) \in F$.

Since, *if for all $x, y, z, u \in A$ with $x \leqslant y \leqslant z, u$ the conditions* (i)–(iii) *hold*, *then* $(F \cap [x, 1]; x \in A) \in \mathrm{CFF}(\mathscr{S}(\mathscr{A}))$ *and according to Theorem* 7 *there exists a* $G \in \mathrm{CK}(\mathscr{A})$ *with* $G \cap [x, 1] = F \cap [x, 1]$ *for all* $x \in A$ *and hence*

$$\begin{aligned} F &= F \cap A = F \cap \bigcup_{x \in A} [x, 1] = \bigcup_{x \in A} (F \cap [x, 1]) = \bigcup_{x \in A} (G \cap [x, 1]) = G \cap \bigcup_{x \in A} [x, 1] \\ &= G \cap A = G \in \mathrm{CK}(\mathscr{A}). \end{aligned}$$

Example 1. Consider the orthomodular lattice $(L, \vee, \wedge, {}', 0, 1)$ with the Hasse diagram



define $x^{0} := x'$ for all $x \in \{a, b, c, d, e\}$, $(a')^{c} := e'$ and $(b')^{c} := d'$ and y^{x} for $x, y \in L$ with $x \leqslant y \leqslant 1$ in all the other cases in the unique possible way such that $([x, 1], \vee, \wedge, {}^{x}, x, 1)$ becomes an orthomodular lattice and put $xy := (x \vee y)^{y}$ for all $x, y \in L$ and $F := \{c', 1\}$. Then $(L, \vee, ({}^{x}; x \in A), 1)$ is an orthomodular join-semilattice and $\mathscr{A} := (L, \cdot, 1)$ its corresponding orthomodular implication algebra. Since $0 \leqslant c \leqslant b'$ and $c^{0} = c' \in F$, but

$$((b')^{0} \wedge (b')^{c}) \vee (b' \wedge ((b')^{c})^{0}) = (b \wedge d') \vee (b' \wedge d) = 0 \vee 0 = 0 \notin F,$$

F does not satisfy condition (ii) of Corollary 2 and hence cannot be a congruence kernel of $\mathscr{A}$, but F satisfies conditions (i) and (iii) of Corollary 2 proving the independence of condition (ii) from conditions (i) and (iii) in Corollary 2.

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